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On trapped waves over a continental shelf

By JOHN M. HUTHNANCE
Department of Oceanography, University of Liverpool

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Any straight continental shelf of monotonic depth profile is shown to have as its entire complement of barotropic trapped modes (i) an infinite discrete set of 'continental-shelf waves', (ii) a single 'Kelvin wave', and (iii) an infinite discrete set of 'edge waves'. The decomposition of energy density and fluxes into modal constituents is discussed.

1. Introduction

In an unbounded ocean of uniform depth, the only free barotropic wave modes with horizontal scales much greater than the depth are plane gravity waves of frequency σ greater than the Coriolis frequency f. However, the presence of a coastline and non-uniform depth along a continental shelf introduces the following theoretical possibilities.

(i) A Kelvin wave. This has no offshore nodes, and in its simplest form in water of uniform depth against a straight coast (Lamb 1932, §208) progresses with the speed of long gravity waves, decaying seawards exponentially.

(ii) Edge waves. This term will be used here for the modes of frequency σ greater than f, and offshore wavenumbers 1, 2, ..., found by Eckart (1951), which depend on the offshore increase in water depth.

(iii) Continental-shelf waves. Discovered by Robinson (1964) for a narrow uniformly sloping model shelf, these modes have offshore wavenumbers 1, 2, ..., frequencies σ less than f and progress along the shelf in a cyclonic sense about the deep sea.

There is now considerable observational evidence for the existence of continental-shelf and edge waves trapped along the coast, beginning (for shelf waves) with the measurements of Hamon (1962) around the Australian coast. A fuller list is given in Huthnance (1973). Laboratory experiments (Caldwell, Cutchin & Longuet-Higgins 1972; Bowen & Inman 1971) confirm the existence of such waves.

There is also an abundance of specific theoretical analyses (see Huthnance 1973) based on particular depth profiles and various assumptions, especially of a narrow shelf and (prompted by applications with atmospheric forcing) low frequencies ($\sigma \ll f$) in the case of continental-shelf waves. A principal aim of the present study is to determine to what extent typical conclusions in specific cases (e.g. that shelf waves always travel with land on their right in the northern hemisphere) apply to arbitrary monotonic profiles and the full range of parameters, and hence to sketch a figure showing the distribution of all trapped modes analogous to that obtained numerically by Munk, Snodgrass & Wimbush (1970, figure 6) for a step-shelf model.

The difficulty with this problem, which has led to the wide variety of solutions for particular cases and under special assumptions, is the fact that the wave frequency, which one generally wishes to regard as the eigenvalue of the system of equations determining trapped modes, appears in two terms (with powers -1 and 2) in the differential equation and also in the boundary conditions (see equations (2.3) and (2.4) below). Alternatively, for tidally forced problems the frequency is fixed and we should like to regard the longshore wavenumber as the eigenvalue. This also appears in two terms (with powers 1 and 2) in the differential equation as well as in the boundary conditions.

The equations of motion and associated eigenvalue problem for trapped modes are set out in §2. Section 3 demonstrates the existence of dispersion relations uniquely determining the frequencies of an infinite discrete set of shelf waves, a single Kelvin wave and an infinite discrete set of edge waves. The involved appearance of the eigenvalue (the frequency σ) in the problem therefore does not introduce any variations in the qualitative distribution of eigenmodes. However, the use of special examples in §4 demonstrates that the magnitude and even the sense of the group velocity are highly profile dependent, with few general statements possible. This has the consequence that for a tide-related problem, where the frequency is specified, there is an unknown (i.e. profile-dependent) number of eigenvalues (i.e. longshore wavenumbers) and corresponding modes.

The nonlinear appearance of the eigenvalue in the differential equation and boundary conditions also leads to orthogonality relations between modes whose form is mode dependent and whose interpretation is not clear. Their connexion with energy flux relations is discussed in §5, where it is shown that only a certain combination of the energy density and flux can be separated into modal constituents.

2. Equations

We use the 'long-wave' equations governing barotropic 'hydrostatic' motion of an inviscid, homogeneous, incompressible sea (with surface elevation ζ above the equilibrium level $z = 0$) overlying the sea floor $z = -h(\mathbf{x})$. Thus

$$\mathbf{u}_t + \mathbf{k} \wedge \mathbf{u} = -\nabla\zeta, \tag{2.1}$$

$$D^2\zeta_t + \nabla . [h\mathbf{u}] = 0, \tag{2.2}$$

where $\mathbf{u} \equiv (u, v)$ is the horizontal velocity vector, $\mathbf{k}$ the unit vector in the direction of z increasing (vertically up) and ∇ is the horizontal gradient operator (∂_x, ∂_y). $\mathbf{x} \equiv (x, y)$ are co-ordinates in the horizontal plane; we have made the 'f-plane' approximation, taking f to be uniform and neglecting the earth's curvature. The quantities $\mathbf{u}$, t, $\mathbf{x}$, ζ and h have been non-dimensionalized on the scales U, f^{-1}, L, fUL/g and H respectively, and the non-dimensional parameter $D^2 \equiv f^2L^2/gH$ compares the shelf-breadth scale L with a Rossby radius of deformation $(gH)^{\frac{1}{2}}/f$ for barotropic motion.

We shall be primarily concerned with a straight shelf, coastline $x = 0$, for which the depth h is independent of the longshore co-ordinate y and $h(x)$ increases monotonically to 1 as x increases towards the deep sea. For natural modes of the form $u(x)\exp i(ky+\sigma t)$, etc., (2.1) enables expression of (u, v) in terms of ζ, which by (2.2) satisfies

$$(h\zeta')' - K\zeta = 0 \quad (0 \leqslant x < \infty), \quad K = -k\sigma^{-1}h' + k^2h + D^2(1-\sigma^2), \tag{2.3}$$

where a prime indicates d/dx. The boundary conditions $hu \to 0$ $(x \to 0)$ and $\zeta \to 0$ $(x \to \infty)$ for trapped waves become

$$h(\zeta' + k\sigma^{-1}\zeta) \to 0 \quad (x \to 0), \tag{2.4}$$

$$\zeta \to 0 \quad (x \to \infty). \tag{2.5}$$

Equations (2.3)–(2.5) form the eigenvalue problem determining the existence of trapped waves. For a given profile shape h, there are three non-dimensional parameters, D, σ and k, representing shelf breadth, mode frequency and longshore wavenumber respectively. The system is of conventional Sturm–Liouville form only if we regard D^2 as the eigenvalue, with σ and k given. However, in any real context we should normally regard the shelf width, and hence D^2, as given, and seek either σ or k as an eigenvalue.

For definiteness, we assume $f > 0$, corresponding to the Northern Hemisphere, and $\sigma > 0$ by convention. Then $k > 0$ corresponds to propagation to the left as viewed from the deep sea (i.e. cyclonically about the deep sea).

3. The set of trapped modes

Noting that all coefficients in (2.3)–(2.5) are real, we assume without loss of generality in the following that ζ is real.

All studies of continental-shelf waves have concluded that (in the Northern Hemisphere) they always progress to the left as viewed from the deep sea. Longuet-Higgins (1968) proved this result for double Kelvin waves over any monotonic profile, and we now show in the present context the following.

(*a*) *Any trapped mode of frequency* $\sigma < 1$ (the inertial frequency) *propagates cyclonically relative to the deep sea* (i.e. $k > 0$).

Proof. $\int_0^\infty \zeta(2.3)\,dx$ yields, after integration by parts and use of (2.4) and (2.5),

$$k\sigma^{-1}J = I_1 + E^2I_2, \tag{3.1}$$

where

$$I_1 = \int_0^\infty h(\zeta'^2 + k^2\zeta^2)\,dx > 0, \quad I_2 = \int_0^\infty \zeta^2 dx > 0, \quad E^2 = D^2(1-\sigma^2) > 0$$

and

$$J = [h\zeta^2]_{x=0} + \int_0^\infty h'\zeta^2 dx > 0. \tag{3.2}$$

Hence $k > 0$.

We now suppose (for analytical purposes only) that σ and k are given (but arbitrary). Then standard Sturm–Liouville theory (Hille 1969, chap. 8) implies that there is a discrete sequence of unique eigenvalues $-E_0^2$ (possibly absent) $< -E_1^2 < \ldots < -E_n^2$ corresponding to eigensolutions of (2.3) with $0, 1, 2, \ldots, n$ nodes (together with a continuum of untrapped modes in $-E^2 > k^2$). The values

E_i^2 depend continuously on k and σ except at $\sigma = 0$. In other words, we have in σ, k, D^2 space a discrete set of continuous (except at $\sigma = 0, 1$) non-intersecting sheets $D^2(\sigma, k)$, each corresponding to a particular mode number. The physical basis of (2.3)–(2.5) restricts our attention to $\sigma > 0$, $D^2 > 0$ (although this renders the set of eigenfunctions mathematically incomplete; Lindzen 1966). According to (*a*), none of the sheets enters $k < 0$, $\sigma < 1$ within our restricted space.

We now confine our attention to one such sheet and suppose k given, so that we consider the form of the function $D^2(\sigma)$ for a given mode. It is shown in appendix A that

(*b*) *D^2 is a monotonic decreasing function of σ^2.*

This is perhaps the most significant result of this section, since it permits us to invert the function $D^2(\sigma)$ on either side of $\sigma = 1$. Thus, for each mode, the sheet in σ, k, D^2 space represents a unique function $\sigma(k, D^2)$. This function may not be defined over the whole range of k and positive D^2; for example, the sheets corresponding to continental-shelf waves do not extend into $k < 0$. However, the uniqueness does imply the following for given k and D^2.

(*c*) *In $\sigma < 1$ (if $k > 0$), there is a discrete sequence of unique modes with normal-to-shore wavenumbers* 0 (possibly absent), 1, 2, ... *and corresponding unique frequencies $\sigma_0 > \sigma_1 > \sigma_2 > \ldots$. In $\sigma > 1$, there is a discrete sequence of unique modes with normal-to-shore wavenumbers* 0 (possibly absent), 1, 2, ... *and corresponding unique frequencies $\sigma_0 < \sigma_1 < \sigma_2 < \ldots$.*

On physical grounds, and by analogy with other physical systems, we should like to regard σ as the eigenvalue of the problem, despite its unorthodox appearance in (2.3)–(2.5). Although the orthogonality relations between the corresponding eigenfunctions remain unsatisfactory, and there is certainly no guarantee that the eigenfunctions form a complete set, (*c*) implies that this is a reasonable point of view. This appears to be simply good fortune; the form of (2.3)–(2.5) suggests that it might have been quite possible for (say) three continental-shelf waves of different frequencies, but all of wavenumber 2, to occur.

We consider only the quadrant $D^2 > 0$, $\sigma > 0$ and (*b*) holds, so that, taking the two sides of $\sigma = 1$ separately, the sheets are 'stacked' in the same order whether regarded as functions $D^2(\sigma)$ or $\sigma(D^2)$. This justifies the inequalities in (*c*), and also implies that all modes exist up to the highest order present. However, we must retain the possibility of only a finite number of modes. In fact, the solutions ζ of (2.3) for large x, where $h = 1$ and $h' = 0$, can exhibit exponential decay (as required for trapping) rather than sinusoidal behaviour (which implies energy fluxes far from the coast) only if

$$\sigma^2 < 1 + k^2/D^2. \tag{3.3}$$

This condition provides a high-frequency limit on the number of modes in $\sigma > 1$ for given k and D^2. There is a continuum of Poincaré waves at higher frequencies. These are not trapped [violating (2.5)], and merely represent that combination of the two linearly independent solutions of (2.3) which satisfies (2.4).

The restriction (3.3) eases for large longshore wavenumbers k. Fixing D and choosing $\sigma = |k|/D$, which satisfies (3.3), we see that by choosing k sufficiently large we can cause K in (2.3) to be arbitrarily large and negative in some non-zero interval (x_1, x_2) over the shelf where $h \leqslant 1-\delta < 1$. Standard oscillation theorems of Sturm–Liouville theory (Hille 1969) therefore imply arbitrarily large normal-to-shelf wavenumbers. That is,

(*d*) *The sequence of edge-wave modes* [$\sigma > 1$ in (*c*)] *extends to arbitrarily large normal-to-shelf wavenumbers for sufficiently large longshore wavenumbers k.*

However, for any given k, the oscillation theorems imply only a finite number of modes. The greatest wavenumber, given k and D^2, may be found in principle by subtracting unity from the number of zero-crossings of the solution of (2.3) with (2.4) when $\sigma^2 = 1+k^2/D^2$. There is a further guide.

(*e*) *The number of edge-wave modes increases* (*decreases*) *with* D^2 *for* $k > 0$ ($k < 0$).

Proof. We compare the numbers of zero-crossings of the solution of (2.3) with (2.4) when $\sigma^2 = 1+k^2/D^2$ for two values $D_1^2 < D_2^2$ of D^2. Using a variant of Prüfer's technique (Hille 1969), let $q = h\zeta'/\zeta$: at zeros of ζ, q tends to $-\infty$ and recommences decreasing from $+\infty$. We have

$$q' = -ah' - q^2/h - k^2(1-h), \quad q(0) = -ah,$$

where $a = k(1+k^2/D^2)^{-\frac{1}{2}}$: $a_1 \lesseqgtr a_2$ as $k \gtrless 0$. Initially, $q_1 \gtrless q_2$, and the curves $q_1(x)$ and $q_2(x)$ never cross since $q_1 = q_2$ implies $q_1' \gtrless q_2'$. Thus, if $k > 0$, q_2 decreases through $-\infty$ (corresponding to a zero of ζ) at least as often as q_1, and vice versa if $k < 0$. The result follows.

For given k and D^2 we can also cause K in (2.3) to be arbitrarily large and negative over some non-zero interval (x_1, x_2) (in which $h' \geqslant \delta > 0$) through the term $-kh'/\sigma$, by taking σ sufficiently small. The oscillation theorems then imply the following.

(*f*) *The sequence of shelf-wave modes* [$\sigma < 1$ *in* (*c*)] *extends to all normal-to-shelf wavenumbers for all k and* D^2.

We now turn to the modes of normal-to-shelf wavenumber zero, whose existence is yet to be determined. The analysis is carried out in appendix B, with results which may be summarized as follows.

(*g*) *For given k and* D^2, *there is a unique Kelvin wave. In* $k > 0$, *the frequency* σ *decreases smoothly through* 1 *as* D^2 *increases through*

$$2k^3 \int_0^\infty h\, e^{-2kx}\, dx.$$

(*h*) *For given* $|k|$ *and* D^2, *an edge wave or Kelvin wave with* $k < 0$ *has a greater frequency* σ *than the corresponding mode with* $k > 0$.

Proof. Let the frequencies of the modes with $k \gtrless 0$ be σ_1 and σ_2,

$$a_{1,2} = \pm |k|(1+k^2/D^2)^{-\frac{1}{2}}, \quad q = h\zeta'/\zeta,$$

where, as in (*e*),

$$q' = -ah' - q^2/h - k^2(1-h), \quad q(0) = -ah.$$

Since $a_1 > a_2$, q_1 (corresponding to a_1) is less than q_2 at $x = 0$, and $q_1 = q_2$ implies that $q_1' < q_2'$, so that q_1 and q_2 cannot cross or meet. Hence, after an equal number of nodes of ζ (where q decreases to $-\infty$ and reappears at $+\infty$ with negative gradient) $q_2 > q_1$, that is

$$-[k^2 + D^2(1-\sigma_2^2)]^{\frac{1}{2}} = q_2(\infty) > q_1(\infty) = -[k^2 + D^2(1-\sigma_1^2)]^{\frac{1}{2}}, \quad \text{i.e.} \quad \sigma_2 > \sigma_1.$$

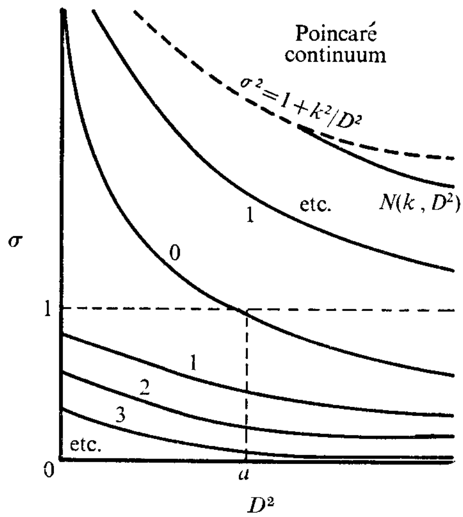

FIGURE 1. A typical dispersion relation $\sigma(k, D^2)$ for given k. Mode numbers equal the numbers of offshore nodes.

$$a = 2k^3 \int_0^\infty h e^{-2kx} dx.$$

The results of this section enable us to sketch σ as a function of D^2 for given $k > 0$ (figure 1). The qualitative features are independent of k except that for $k < 0$ there are no modes in $\sigma < 1$ and the number of edge waves decreases with increasing D^2.

4. Group velocity

The results of § 3 essentially demonstrate the distribution of natural modes in the D^2, σ plane. While this has value in enabling a count of all relevant modes in many contexts, the nature of the frequency σ as a function of the longshore wavenumber k for any given mode is often of more physical interest. In particular, the slope of this function (the group velocity c_g) indicates the sense and speed of the expected energy propagation. For problems where the frequency is imposed externally, such as the forcing of waves over the shelf by the tides, we also need the form of this function to determine the corresponding wavelengths for the natural modes, and whether (for example) a shelf wave with two offshore nodes could have two corresponding wavelengths. Unfortunately, the function $\sigma(k)$ for given D^2 is strongly profile dependent, and we have only three general results.

(*i*) *If* $k > 0$, *the number of edge-wave modes increases with* k.

The proof is essentially the same as for (*e*).

(*j*) *If* $k < 0$, *then* $c_g < 0$, i.e. *the phase and group velocities are in the same sense.*

Proof. We first recall that a given mode defines a dispersion relation

$$\sigma = \sigma(k, D^2),$$

so that
$$\left.\frac{\partial\sigma}{\partial k}\right|_{D^2} = -\left.\frac{\partial\sigma}{\partial D^2}\right|_k \left.\frac{\partial D^2}{\partial k}\right|_\sigma.$$
Since $\partial\sigma/\partial D^2 < 0$ [result (b)], the group velocity has the same sign as $\partial D^2/\partial k$. If we allow a small change δk in k (holding σ fixed), with corresponding small changes δD^2 in D^2 (i.e. δE^2 in E^2) and $\delta\zeta$ in the eigenfunction ζ, then (3.1) yields, after use of (2.3)–(2.5),
$$\left.\frac{\partial E^2}{\partial k}\right|_\sigma = \frac{J/\sigma - 2kI_3}{I_2} > 0,$$
where
$$I_3 = \int_0^\infty h\zeta^2 dx > 0.$$
Thus
$$\left.\frac{\partial D^2}{\partial k}\right|_\sigma = \frac{1}{1-\sigma^2}\left.\frac{\partial E^2}{\partial k}\right|_\sigma < 0$$
and the result follows.

(k) *In* $\sigma < 1 (k > 0)$, $\sigma \leqslant k/D$. *As* $k \to 0$, $c_g \sim \sigma/k > 0$. *If* h'/h *is bounded, all shelf waves have* $c_g < 0$ *for some* k.

Proof. Define
$$p \equiv \frac{h\zeta'}{\zeta} + \frac{kh}{\sigma}, \quad p_\pm(h) \equiv h\left\{\frac{k}{\sigma} \pm \left[k^2 + \frac{D^2}{h}(1-\sigma^2)\right]^{\frac{1}{2}}\right\} \quad \text{(real)}.$$
If $p_-(h) < p < p_+(h)$, then $p' > 0$ (see appendix B). If $p_-(h) < 0$ for all $h \leqslant 1$, then $p = 0$ leads to $p' > 0$ since $p_+(h) > 0$, so that $p(\infty) \geqslant 0 > p_-(1)$ since $p(0) = 0$ (coastal boundary condition), and no mode is possible (we require $p(\infty) = p_-(1)$). Hence, for some h, $p_-(h) \geqslant 0$, i.e. $\sigma \leqslant kD^{-1}h^{\frac{1}{2}} \leqslant k/D$.

From (3.1),
$$\left.\frac{\partial\sigma}{\partial k}\right|_{D^2} = \frac{J/\sigma - 2kI_3}{kJ/\sigma^2 - 2D^2\sigma I_2} \quad \text{[cf. proof of } (j)]$$
$$= \frac{(I_1 + E^2 I_2)/k - 2kI_3}{(I_1 + E^2 I_2)/\sigma - 2D^2\sigma I_2} \quad \text{by (3.1)}$$
$$\sim \sigma/k \quad \text{as} \quad k \to 0 \quad \text{(which implies that} \quad \sigma \to 0).$$

For all modes with a node we require in (2.3) that $K < 0$ somewhere, hence somewhere $kh'/\sigma > k^2 h$ for shelf waves ($\sigma^2 < 1$). Thus $\sigma < k^{-1}\sup(h'/h) \to 0$ as $k \to \infty$. If $\sigma = \sigma_0$ at $k = k_0$, then $\sigma < \sigma_0$ eventually as $k \to \infty$, and so, for some k, $c_g < 0$.

However, some condition playing the role of 'h'/h bounded' is necessary in (k) for the following reason.

(l) *Shelf waves for the step-shelf profile*
$$h = \begin{cases} \Delta & (0 < x < 1) \\ 1 & (x > 1) \end{cases}$$
always have positive group velocity. (The phase velocity is positive, i.e. $k > 0$, by result (a).)

Proof. Owing to the concentration of the depth change at one point with this profile, the only shelf waves have just one offshore node: the dispersion relation is
$$\left(\frac{\Delta}{\sigma^2} - X\right)(1-\sigma^2) = \left\{\frac{1}{\sigma} - (1+X)^{\frac{1}{2}}\right\}\left\{\frac{1}{\sigma} - \left(1 + \frac{X}{\Delta}\right)^{\frac{1}{2}} \coth\left[k\left(1 + \frac{X}{\Delta}\right)^{\frac{1}{2}}\right]\right\}, \quad (4.1)$$

where $X = D^2/k^2$ and all factors in (4.1) are positive. Their behaviour as X increases from X_0 (for which (4.1) holds) is that

$$F \equiv \Delta/\sigma^2 - X \quad \text{decreases linearly to zero at} \quad X = \Delta/\sigma^2.$$

$$G \equiv \frac{1}{\sigma} - \left(1+\frac{X}{\Delta}\right)^{\frac{1}{2}} \coth\left[k\left(1+\frac{X}{\Delta}\right)^{\frac{1}{2}}\right] \quad \text{becomes zero before} \quad X = \frac{\Delta}{\sigma^2} - \Delta.$$

It may be verified that $\partial^2 G/\partial X^2 > 0$, so that, at X_0,

$$\begin{aligned}\frac{\partial}{\partial X}[\mathrm{RHS}(4.1)] &< \left(\frac{1}{\sigma} - (1+X_0)^{\frac{1}{2}}\right)\frac{\partial G}{\partial X} < \left(\frac{1}{\sigma} - (1+X_0)^{\frac{1}{2}}\right)\left[\frac{-G}{\Delta/\sigma^2 - \Delta - X_0}\right] \\ &< -(1-\sigma^2)\frac{F}{\Delta/\sigma^2 - X_0} \quad [\text{using}\,(4.1)] \\ &= \frac{\partial}{\partial X}[\mathrm{LHS}\,(4.1)].\end{aligned}$$

Allowing a small change δk in (4.1), holding σ fixed, with a corresponding change δX, yields

$$\delta X \frac{\partial}{\partial X}[\mathrm{LHS}\,(4.1) - \mathrm{RHS}(4.1)] = \frac{[\sigma^{-1} - (1+X)^{\frac{1}{2}}](1+X/\Delta)}{\sinh^2[k(1+X/\Delta)^{\frac{1}{2}}]}\delta k.$$

Thus $[\partial X/\partial k]_\sigma > 0$. Hence $[\partial D^2/\partial k]_\sigma > 0$ and $c_g > 0$ [see proof of (j)].

It therefore appears that the vertical cliff in the step profile is responsible for anomalous frequencies σ at high longshore wavenumbers k. Whereas 'most' profiles (for example, the Buchwald & Adams (1968) exponential profile) predict shelf-wave frequency maxima at an intermediate longshore wavenumber, corresponding to vanishing group velocity, and a frequency decrease to zero at large longshore wavenumbers, the step-model frequency increases monotonically towards the limit $(1-\Delta)/(1+\Delta)$ at large longshore wavenumbers. We can understand the anomaly by noting that for shelf waves the restoring force depends essentially on the depth change 'seen' by the wave. For the step model (and others including a cliff), this remains large while the normal-to-shore extent of the wave decreases with the inverse longshore wavenumber, but in general for continuous depth profiles the depth change decreases correspondingly. The anomalous behaviour of the step shelf gives it the advantage, when discussing modes at specified tidal frequencies, of assuring just one free shelf wave (Munk *et al.* 1970).

Edge waves ($\sigma^2 > 1$) have generally been found in examples to have $c_g > 0$ when $k > 0$. However, this is not always the case. As a counterexample, we have the second-mode (two nodes) edge wave for the profile

$$h = \begin{cases} 10^{-4}, & x < 1, \\ 1, & x > 1, \end{cases}$$

with $D^2 = 0{\cdot}000174$. The frequencies $\sigma = 1{\cdot}5005 \pm 10^{-4}$ and $1{\cdot}4998 \pm 10^{-4}$ correspond to $k = 0{\cdot}45$ and $0{\cdot}6$. The existence of this example, extreme as it is, precludes the possibility of any qualitative results. In the absence of rotation, of course, the group velocity for edge waves is always in the sense of phase propagation.

Figure 2 is a typical but not universal sketch of the dispersion curves $\sigma(k)$ for given D^2.

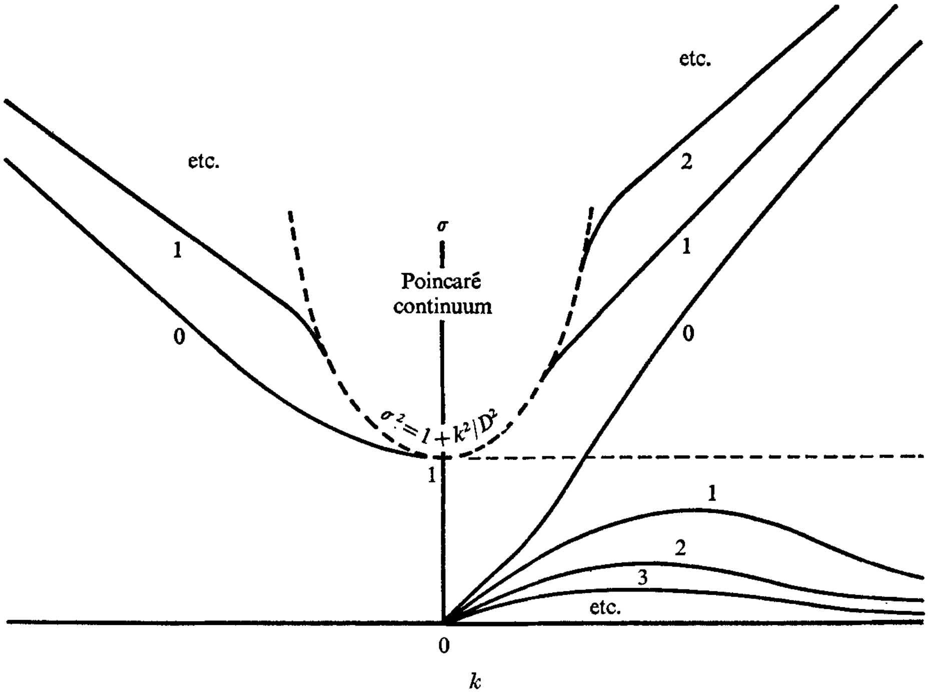

FIGURE 2. A typical dispersion relation $\sigma(k, D^2)$ for given D^2. Mode numbers equal the numbers of offshore nodes.

5. Energy flux and orthogonality

For two trapped-mode solutions (ζ_1, k_1, σ_1) and (ζ_2, k_2, σ_2) of (2.3) in a specified context (h, D^2), $\int \{\zeta_1(2.3)_2 - \zeta_2(2.3)_1\} dx$ gives an orthogonality relation between ζ_1 and ζ_2. However, its form depends on the particular modes. If we require the modes to be members of the sequences (i) σ specified, e.g. tidal forcing, (ii) k specified, e.g. shelves around an island for which the longshelf wavenumber is quantized, or (iii) phase velocity c specified, e.g. forcing by an atmospheric disturbance travelling along the shelf, then this relation simplifies somewhat. Only in case (iii), however, does it become mode independent. Thus it is possible to solve a problem of type (iii) for the amount of each mode ζ_n generated by taking the scalar product

$$\int_0^\infty (2.3)\, \zeta_n \, dx$$

of (2.3) (now inhomogeneous with forcing term F) with ζ_n. The result is proportional to the constituent

$$\int_0^\infty F \zeta_n \, dx$$

of F. This has been carried out by Gill & Schumann (1974) for the wind-stress forcing of continental-shelf waves. Such a procedure is not possible for problems of types (i) or (ii).

It may be verified that the three orthogonality relations imply respectively the following.

(m, i) *The wave-energy flux*

$$\mathscr{F} = \int_0^\infty h\zeta v\, dx$$

separates into contributions from individual modes (with no contributions from interactions between modes).

(m, ii) *The wave-energy density*

$$\mathscr{E} = \frac{1}{2}\int_0^\infty \{h(u^2+v^2) + D^2\zeta^2\}\, dx,$$

after averaging along the shelf, separates into contributions from individual modes.

(m, iii) $c\mathscr{E}+\mathscr{F}$ *separates into contributions from individual modes* (N.B. the sign convention followed gives negative $c = \sigma/k$ for propagation in the sense of y increasing).

However, the Buchwald & Adams (1968) exponential profile provides an example showing that in the three cases no other linear combination of $\mathscr{E}$ and $\mathscr{F}$ separates thus.

Results (m) may be summarized symbolically by

$$`c_g\text{'}\,\mathscr{E}_{nm} + \mathscr{F}_{nm} = 0,$$

where $\mathscr{E}_{nm}$ and $\mathscr{F}_{nm}$ are the contributions to wave-energy density and flux respectively from interactions between different modes m and n, and

$$`c_g\text{'} \equiv \frac{\sigma_n - \sigma_m}{k_n - k_m} = \left\{\begin{matrix} 0 & \text{in case (i)}, \\ \infty & \text{in case (ii)}, \\ c & \text{in case (iii)}, \end{matrix}\right\} \quad \text{independently of } m, n,$$

is merely a suggestive notation. This equation and the fact that the orthogonality relations and the energy conservation equation are both quadratic in the wave motion suggest that the two are related.

To clarify this we consider the energy conservation relation

$$\partial\mathscr{E}/\partial t + \partial\mathscr{F}/\partial y = 0,$$

which may be found from

$$\int_0^\infty \{h\mathbf{u}\,.\,(2.1) + \zeta(2.2)\}\, dx.$$

By considering the form of this when just two wave modes m and n are present with various amplitudes, we find

$$\partial\mathscr{E}_{mn}/\partial t + \partial\mathscr{F}_{mn}/\partial y = 0.$$

Since the solutions ζ_m and ζ_n of (2.3) are real, we may write

$$\mathscr{E}_{mn} = \mathscr{E}^s_{mn} \sin\phi_m \sin\phi_n + \mathscr{E}^c_{mn}\cos\phi_m\cos\phi_n, \quad \mathscr{F}_{mn} = \mathscr{F}^c_{mn}\cos\phi_m\cos\phi_n,$$

where

$$\phi = ky + \sigma t.$$

Hence

$$0 = \left\{\frac{\sigma_n+\sigma_m}{2}(\mathscr{E}^s_{mn} - \mathscr{E}^c_{mn}) - \frac{k_m+k_n}{2}\mathscr{F}^c_{mn}\right\}\sin(\phi_m+\phi_n)$$
$$+\left\{\frac{\sigma_m-\sigma_n}{2}(\mathscr{E}^s_{mn} + \mathscr{E}^c_{mn}) + \frac{k_m-k_n}{2}\mathscr{F}^c_{mn}\right\}\sin(\phi_n-\phi_m).$$

The two terms in braces must be separately zero, so that in the three cases we obtain respectively

(i) (m, i) and $\int_0^\infty h u_n u_m \, dx = \int_0^\infty (h v_n v_m + D^2 \zeta_n \zeta_m) \, dx,$

(ii) (m, ii) and $$0 = \frac{\sigma_n + \sigma_m}{k} \int_0^\infty (h v_n v_m + D^2 \zeta_n \zeta_m) \, dx + \int_0^\infty h(\zeta_n v_m + \zeta_m v_n) \, dx,$$

(iii) (m, iii) and $0 = \int_0^\infty h u_n u_m \, dx.$

Thus the physical interpretations [principally results (m)] of the orthogonality relations between modes are simple consequences of energy conservation. We again emphasize that energy $\mathscr{E}$ and flux $\mathscr{F}$ can be separated into modal constituents only in the combinations $\mathscr{F}$, $\mathscr{E}$ and $c\mathscr{E} + \mathscr{F}$ respectively for the cases σ, k and c specified.

6. Conclusions

The existence and uniqueness of the frequency $\sigma(k, D^2)$ corresponding to each member of the full set of trapped modes over a continental shelf have been demonstrated.

Continental-shelf waves have subinertial frequencies, which decrease (to zero) with increasing mode numbers corresponding to 1, 2, ... offshore nodes (this is an infinite set). Phase propagation is always cyclonic relative to deep water, but the group velocity is always in the reverse sense for some (typically all) sufficiently large longshore wavenumbers (with the exception of profiles having unbounded h'/h: the shelf wave for the step-shelf model never has reversed group velocity). The frequency also tends to zero for small longshore wavenumbers, so that there is a maximum frequency at which the group velocity vanishes. Since [cf. proof of (k)]

$$c_g = \frac{J - 2k\sigma I_3}{k\sigma^{-1} J - 2D^2\sigma^2 I_2} = \frac{\mathscr{F}}{\mathscr{E}}$$

by straightforward calculation (where $\mathscr{F}$ and $\mathscr{E}$ are time-averaged), wave energy at this frequency cannot propagate along the shelf.

Edge waves may travel along the shelf in either direction. Apart from the fundamental mode (no offshore nodes) travelling cyclonically relative to deep water, which has subinertial frequencies at low wavenumbers,

$$2k^3 \int_0^\infty h e^{-2kx} \, dx < D^2,$$

they have superinertial frequencies which increase with increasing normal-to-shore wavenumber $(= 0, 1, 2, \ldots, n(k, D^2))$ up to a limit $\sigma^2 = 1 + k^2/D^2$, which is the trapping condition. An arbitrarily large number of modes occurs for sufficiently large longshore wavenumbers k. Edge waves travelling cyclonically relative to the deep water have lower frequencies than those travelling anticyclonically.

It appears that the set of trapped modes divides naturally into two classes, each associated with just one of the two restoring forces present (potential-vorticity conservation for continental-shelf waves and gravity for edge waves). The two combine in the fundamental edge wave travelling cyclonically about the deep sea; this exhibits a transition from one class to the other for increasing longshore wavenumbers. Apparently the two mechanisms interact only in making possible a reverse group velocity for cyclonically travelling edge waves.

However, the presence of both mechanisms renders the orthogonality relations between trapped modes mode-dependent, unless they are members of a sequence with a common value of the phase velocity σ/k. Only in this case may the generation of each mode in a forced oscillation problem be considered independently. Nevertheless, the orthogonality relations remain essentially equivalent to the energy conservation equation in all cases.

We have restricted attention to monotonic depth profiles. The principal effect of a reversal of the bottom slope is to introduce a further infinite set of modes with energy concentrated over the region of reversed slope. These are analogous to the continental-shelf waves already considered, but propagate in the opposite sense.

The analysis of §§3–5 applies without substantial modification to systems with circular depth contours replacing the straight shelf. The Kelvin wave of azimuthal wavenumber n (an integer) crosses $\sigma = 1$ at

$$D^2 = \begin{cases} \dfrac{2n^2(n-1)}{r_0^2} \displaystyle\int_{r_0}^{\infty} h \left(\frac{r_0}{r}\right)^{2n+1} \frac{dr}{r_0} & (\text{island of radius } r_0\text{: } n > 0), \\ \dfrac{2n^2(1-n)}{r_0^2} \displaystyle\int_{0}^{r_0} \left(\frac{r}{r_0}\right)^{-2n-1} h \frac{dr}{r_0} & (\text{basin of radius } r_0\text{: } n < 0). \end{cases}$$

The basic result (*b*) may be shown to hold also (cf. appendix A) for an oceanic ridge, so that again in this case there follows a simple classification of trapped modes into double Kelvin waves conserving potential vorticity, travelling over the lateral slopes anticyclonically relative to the shallow region over the ridge (Longuet-Higgins 1968), and gravity waves (analogues to edge waves) trapped over the ridge with superinertial frequencies (Buchwald 1969).

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Appendix A. Proof of (*b*)

(i) $0 < \sigma < 1$, $k > 0$

In this case the existence of $D^2(\sigma^2)$ permits definition of $E^2(\sigma^2) \equiv (1-\sigma^2)\, D^2(\sigma^2)$, which is positive in the range of interest. If we differentiate (3.1) with respect to σ^2 (holding k fixed), allowing variations in E^2 and in the eigenfunction ζ, then we obtain, after use of (2.3)–(2.5),

$$\partial E^2/\partial\sigma^2 = -kJ/2\sigma^3 I_2 < 0.$$

Hence we can invert $E^2(\sigma^2)$ into a monotonically decreasing function $\sigma^2 = F(E^2)$. Now

$$\frac{dD^2}{d\sigma^2} = \frac{d}{d\sigma^2}\frac{F^{-1}(\sigma^2)}{1-\sigma^2} = \frac{1-F+E^2\,dF/dE^2}{(dF/dE^2)(1-\sigma^2)^2},$$

so that $dD^2/d\sigma^2 < 0$ if and only if $F-E^2dF/dE^2 < 1$, i.e. the tangent to $F(E^2)$ interests the F axis below 1 (figure 3*a*).

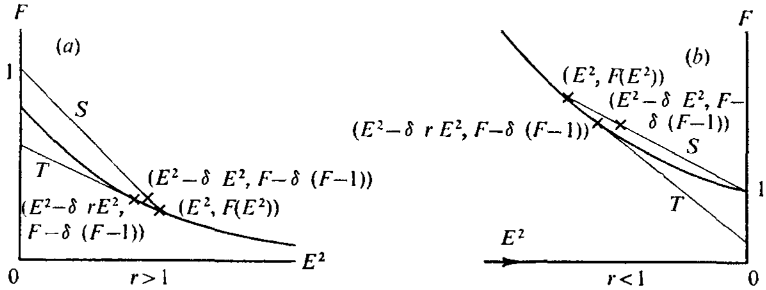

FIGURE 3. Sketches for appendix A. S is the straight line between $(E^2,\ F(E^2))$ and $(0,\ 1)$. T is the tangent at $(E^2,\ F(E^2))$.

We now show that the straight line between $(E^2,\ F(E^2))$ and $(0, 1)$ is always above the curve for abscissae just less than E^2, from which the result follows.

Proof. Suppose that $(E^2-rE^2\delta,\ F-\delta(F-1))$ is on the curve as $\delta\to 0$, with corresponding eigenfunction χ:

$$(h\chi')'-L\chi = 0,\quad L = -kh'/[F-\delta(F-1)]^{\frac{1}{2}}+k^2h+E^2-rE^2\delta, \tag{A 1}$$

$$h\{\chi'+k[F-\delta(F-1)]^{-\frac{1}{2}}\chi\} = 0 \quad (x=0),\quad \chi\to 0 \quad (x\to\infty). \tag{A 2}$$

Then

$$\int_0^\infty [\zeta(\mathrm{A}\,1)-\chi(2.3)]\,dx$$

gives, after using the boundary conditions (2.4), (2.5) and (A 2),

$$\begin{aligned} 0 &= -rE^2\int_0^\infty \zeta\chi\,dx+\frac{k}{\delta\sqrt{F}}\left(1-\frac{1}{(1-\delta+\delta/F)^{\frac{1}{2}}}\right)\left\{\int_0^\infty h'\zeta\chi\,dx+h\zeta\chi|_0\right\} \\ &= -rE^2I_2+\frac{k}{2\sqrt{F}}\left(\frac{1}{F}-1\right)J+O(\delta) \quad (\delta\to 0;\ \text{ i.e. }\ \chi\to\zeta). \end{aligned}$$

Substituting for E^2I_2 from (3.1) yields

$$\begin{aligned} \frac{k}{2\sqrt{F}}\left(2+\frac{1}{r}-\frac{1}{Fr}\right) = \frac{I_1}{J} &= \left|\frac{\int_0^\infty (h\zeta'^2+hk^2\zeta^2)\,dx}{h\zeta^2|_0+\int_0^\infty h'\zeta^2\,dx}\right| = \left|\frac{\int_0^\infty (h\zeta'^2+hk^2\zeta^2)\,dx}{-2\int_0^\infty h\zeta\zeta'\,dx}\right| \\ &= k\left|\frac{\int_0^\infty [(\zeta'\sqrt{h})^2+(k\zeta\sqrt{h})^2]\,dx}{2\int_0^\infty (\zeta'\sqrt{h})(k\zeta\sqrt{h})\,dx}\right| \geqslant k \end{aligned} \tag{A 3}$$

by the Cauchy–Schwarz inequality.

If $r \leqslant 1$, the greatest value of the left side of the inequality in the range $0 \leqslant F \leqslant 1$ is k and is attained at $F = 1$, not in $F < 1$. Hence $r > 1$, i.e. the straight line is above the curve point $(E - rE^2\delta,\ F - (F-1)\delta)$ as required (figure 3*a*).

(ii) $\sigma > 1,\ k > 0$

The proof follows that of (i) verbatim as far as (A 3), except that E^2 is negative and the point $(E^2 - rE^2\delta,\ F - \delta(F-1))$ has abscissa greater than E^2.

If $r \geqslant 1$, the greatest value of the left side of (A 3) in the range $F \geqslant 1$ is k and is attained at $F = 1$, not in $F > 1$. Hence $r < 1$, i.e. the straight line is above the curve point $(E - rE^2\delta,\ F - \delta(F-1))$ as required (figure 3*b*).

(iii) $\sigma > 1,\ k < 0$

In this case we have [cf. (i)] $E^2(\sigma^2) < 0$ and $dE^2/d\sigma^2 > 0$. Hence

$$\frac{dD^2}{d\sigma^2} = \frac{d}{d\sigma^2}\frac{E^2}{1-\sigma^2} = \frac{(1-\sigma^2)\,dE^2/d\sigma^2 + E^2}{(1-\sigma^2)^2} < 0.$$

Appendix B. The existence of a Kelvin wave, (*g*)

Let $p = h\zeta'/\zeta + kh/\sigma$; zeros of ζ correspond to singularities of p. Kelvin-wave solutions ζ, having no offshore nodes, correspond to functions p with a finite value everywhere.

Equations (2.3)–(2.5) become

$$p' = -(p - p_+(h))(p - p_-(h))/h, \quad p(0) = 0, \quad p(\infty) = p_-(1),$$

where
$$p_\pm(h) = h\left\{\frac{k}{\sigma} \pm \left[k^2 + \frac{D^2}{h}(1-\sigma^2)\right]^{\frac{1}{2}}\right\}.$$

Thus p' is negative unless p_- and p_+ are real and $p_- < p < p_+$. p can become infinite only by decreasing to $-\infty$ and 'reappearing' with value $+\infty$ and $p' < 0$.

If we integrate the differential equation for p out from $x = 0$, where $p = 0$, to X, the result $p(X)$ depends continuously on the parameters k, σ and D^2 (provided that p does not become infinite), as does $p_-(1)$. Hence if (for example) k and σ are fixed and

$$p(X; D^2 = D_1^2) > p_-(1; D^2 = D_1^2), \quad p(X; D^2 = D_2^2) < p_-(1; D^2 = D_2^2),$$

then for some value D_0^2 between D_1^2 and D_2^2, $p(X; D^2 = D_0^2) = p_-(1; D^2 = D_0^2)$. By taking X sufficiently large (if $h(x) = 1$ for all $x \geqslant x_0$ then $X \geqslant x_0$ suffices) it follows that k, σ and D_0^2 correspond to a Kelvin wave (p being finite). This argument is the basis for demonstrating the existence of a Kelvin wave in all three cases below.

(i) $k < 0 \quad (\sigma \geqslant 1)$

For both $k < 0$ and $k > 0$ we adopt the convention of a positive square root in $p_\pm$ when this is real. Thus (if $k < 0$) $p_-(h) < hk/\sigma\,(< 0) < p_+(h)$. We vary σ (rather than D^2 as above). Let $\sigma_1 = 1$; then $p_+ = 0$, so that

$$p(x) = 0 > -2|k| = p_-(1)$$

for all x. As σ increases, $p_-(1)$ increases. When $\sigma^2 = \sigma_2^2 \equiv 1 + k^2/D^2$,

$$p_-(1) = k/\sigma_2, \quad p' = -\frac{1}{h}\left(p - \frac{hk}{\sigma_2}\right)^2 - k^2(1-h).$$

Thus $p' < 0$ unless $h = 1$ and $p = k/\sigma_2 = p_-(1)$. Hence, if $\sigma = \sigma_2$, either $p(x)$ decreases through $p_-(1)$ (as x increases) or $p(x) \to p_-(1)$ as $x \to \infty$. In the latter case, $\sigma = \sigma_2$ corresponds to a Kelvin wave. In the former, we have

$$p(X; \sigma = \sigma_2) < p_-(1)$$

for sufficiently large X, and the conditions for establishing a Kelvin wave are met provided that we choose the smallest value $\sigma_0 > \sigma_1$ giving

$$p(X;\, \sigma = \sigma_0) = p_-(1)$$

(to avoid p decreasing to $-\infty$; we know this does not occur for σ near $\sigma_1 = 1$).

(ii) $k > 0$

We vary D^2. Given $\sigma^2 > 1$, if $D^2 = D_1^2 \equiv 0$, then $p_- < 0 < p_+$, so that $p = 0$ implies that $p' > 0$ (for any h). Hence, for all x, $p \geqslant p(0) = 0 > p_-(1)$. If

$$D^2 = D_2^2 \equiv k^2/\sigma^2, \quad \text{then} \quad p_-(h) > 0,$$

so that $p = 0$ implies that $p' < 0$ (for any h). Hence (while p remains finite) $p(x) \leqslant p(0) = 0 < p_-(1)$. The conditions for establishing a Kelvin wave are met provided that we choose the smallest value $D_0 > D_1$.

For $\sigma^2 < 1$ the demonstration is identical with that for $\sigma^2 > 1$ except that

$$D_1^2 \equiv k^2/\sigma^2, \quad D_2^2 \equiv 0.$$

Thus ($k > 0$) we have a Kelvin wave in $\sigma^2 \geqslant 1$ for all D^2 less than some upper limit where $\sigma = 1$ (since σ decreases in D^2), and also in $\sigma^2 \leqslant 1$ for all D^2 greater than some lower limit where $\sigma = 1$. On $\sigma = 1$, we have the unique trapped-wave solution $\zeta = e^{-kx}$. Regarded as a system with the three parameters k, σ and $E^2 \equiv D^2(1-\sigma^2)$, (2.3)–(2.5) depends continuously on σ near $\sigma = 1$, $E^2 = 0$, implying unique Kelvin-wave modes adjacent to the $\sigma = 1$ solution. We must identify these with the unique modes already known in $\sigma^2 \gtrless 1$. Moreover, from (3.1) we have (see appendix A)

$$D^2 = -\frac{dE^2}{d\sigma^2}\bigg|_{\sigma=1} = +\frac{kJ}{2\sigma^3 I_2}\bigg|_{\sigma=1} = 2k^3\int_0^\infty h\, e^{-2kx}\, dx$$

with $dD^2/d\sigma^2$ also continuous across $\sigma = 1$.

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